THE SPECTRUM OF TENSOR-POWER GROWTH FOR THE NATURAL MODULE OF $\mathrm{SL}_2(2^r)$

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Abstract. Let V be the natural module of $\mathrm{SL}_2(q)$ over an algebraically closed field of characteristic 2, where $q = 2^r$. We determine its fusion graph and the complete spectrum of its action matrix. The $2q-1$ vertices are the restrictions of the algebraic-group tilting modules $T(0), \dots, T(q-2)$ and the q projective indecomposable modules. The spectrum is simple, comprising the Brauer character values of V on the projective block and the transient eigenvalues $\{2\cos(j\pi/q) : 1 \le j \le q-1\}$. Thus, if $b(n)$ counts the indecomposable summands of $V^{\otimes n}$ with multiplicity, then for $r \ge 2$ we have

$$b(n) = \frac{3^r}{q(q^2-1)}\,2^n + \frac{S_q}{q+1}\left(-2\cos\frac{\pi}{q+1}\right)^n + O_q\left(\left(2\cos\frac{\pi}{q}\right)^n\right),$$

where $S_q = \prod_{j=0}^{r-1}(1 + 2\cos(2^{j+1}\pi/(q+1)))$ is negative and has degree $\varphi(q+1)/(2r)$ over $\mathbb{Q}$. The two extreme transient eigenvalues $\pm 2\cos(\pi/q)$ both have positive coefficients in the exact expansion; for $q = 2$, the count is instead $b(n) = 2^{n-1}$ for every $n \ge 1$. The argument combines a cyclic tensor-product rule with a weight bound preserving indecomposability under restriction and an elementary determination of the Galois stabilizer of S_q.

1. Introduction

Tensor powers of a modular representation need not be semisimple, and their characters record composition factors rather than the decomposition into indecomposable direct summands. We are interested in the number of these summands. In particular, we ask not only for its exponential growth, but for the error after the leading asymptotic term has been removed; character values alone do not in general determine this error.

For a finite group H and a finite-dimensional kH-module W over an algebraically closed field k, let $b^{H,W}(n)$ be the number of indecomposable summands of $W^{\otimes n}$, counted with multiplicity (we include $W^{\otimes 0} = k$). These indecomposable modules are the vertices of the *fusion graph* of W, whose action matrix is

$$M_{YX} = [W \otimes X : Y]_{\oplus}, \tag{1.1}$$

where the subscript denotes direct-summand multiplicity, not composition multiplicity. We index columns by sources and regard the action matrix as a matrix over $\mathbb{C}$. When the graph is finite, the count is recovered from the vector e_0 of the trivial module by

$$b^{H,W}(n) = \mathbf{1}^{\mathsf{T}} M^n e_0, \tag{1.2}$$

which brings the problem into the spectral theory of nonnegative matrices (see [1, 6, 7, 4]).

For faithful W, He [4, Theorem 1] gives the leading term in terms of the irreducible Brauer characters of H. Let g_t run over the p-regular conjugacy classes, with $p = \operatorname{char} k$, and set

$$S_t = \sum_L \chi_L(g_t^{-1}),$$

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where the sum is over the simple kH-modules. If $Z_W(H)$ is the set of elements acting as scalars on W, and $\omega_W(g) \in \mathbb{C}$ is the lift of that scalar used in Brauer characters, then the leading term is

$$a^{H,W}(n) = \frac{1}{|H|} \sum_{g_t \in Z_W(H)} S_t \omega_W(g_t)^n (\dim W)^n, \tag{1.3}$$

and He obtains, for a constant $c > 0$, the estimate

$$\left| \frac{b^{H,W}(n)}{a^{H,W}(n)} - 1 \right| = O\left(\left| \frac{\lambda^{\mathrm{sec}}}{\dim W} \right|^n + n^{-c} \right). \tag{1.4}$$

Here λ^{sec} is the second spectral scale of the full action matrix, in the sense of [7, Theorem 5.9] (for a finite fusion graph, this is the ordinary second-largest eigenvalue in modulus after removing the peripheral spectrum; see [4, Remark 3]).

The projective vertices form a finite closed component, the *projective cell*, whose eigenvalues are Brauer character values. This component need not determine λ^{sec}: in He's four-dimensional example for $\mathrm{SL}_2(7)$, a subdominant eigenvalue lies outside the $\mathbb{Z}$-span of character values [4, Example 14]. For the natural module of $\mathrm{SL}_2(8)$, He computes a fifteen-vertex graph with a subdominant eigenvalue of modulus approximately 1.8794 [4, Example 15 and Figure 9]. We determine this graph and all its eigenvalues uniformly for the natural module of $\mathrm{SL}_2(2^r)$.

1.1. **Main results.** We work over an algebraically closed field k of characteristic 2, and fix

$$q = 2^r, \qquad r \geq 1, \qquad \Gamma = \mathrm{SL}_2(q) \subset G = \mathrm{SL}_2(k).$$

Let $V = k^2$ be the natural module, and write $b(n)$ for $b^{\Gamma,V}(n)$. Since V is faithful and simple and the only scalar matrix in Γ is the identity, He's leading term reduces to

$$a(n) = \frac{3^r}{q(q^2-1)}\, 2^n. \tag{1.5}$$

For $m \geq 0$, let $T(m)$ denote the indecomposable tilting G-module of highest weight m. Put $R = \mathbb{Z}/r\mathbb{Z}$. The simple $k\Gamma$-modules are

$$L(S) = \bigotimes_{j \in S} V^{[j]}, \qquad S \subseteq R,$$

where $[j]$ denotes the jth Frobenius twist. Let $P(S)$ be the projective cover of $L(S)$. We suppress restriction to Γ when no ambiguity can arise.

Theorem 1.1 (The fusion graph)**.** *The fusion graph of V has exactly $2q-1$ vertices. They are*

$$T(0)|_\Gamma, \ldots, T(q-2)|_\Gamma \quad \text{and} \quad P(S), \qquad S \subseteq R.$$

The first $q-1$ modules are indecomposable, nonprojective, and pairwise nonisomorphic. With these vertices placed before the projective vertices, the action matrix has the form

$$M = \begin{pmatrix} B & 0 \\ C & \widetilde{M} \end{pmatrix}. \tag{1.6}$$

The matrix B is the algebraic-group tilting action matrix truncated to highest weights at most $q-2$. It is block lower triangular, with a diagonal block B_z on

$$\{T(m) : 2^z - 1 \leq m \leq 2^{z+1} - 2\}, \qquad 0 \leq z \leq r-1.$$

This block has size 2^z and characteristic polynomial

$$\det(xI - B_z) = 2\mathsf{T}_{2^z}(x/2), \tag{1.7}$$

where T_d is the Chebyshev polynomial of the first kind. All entries of M are given by the tensor rule in Section 3 and the projective decompositions in Proposition 5.1.

The off-diagonal blocks of B record carries that increase the digit length from z to $z+1$; see Proposition 4.3.

Theorem 1.2 (The spectrum)**.** *The matrix M has simple spectrum and is diagonalizable. Its projective eigenvalues are*

$$2, \qquad 2\cos\frac{2\pi l}{q-1} \quad \left(1 \le l \le \frac{q-2}{2}\right), \qquad 2\cos\frac{2\pi m}{q+1} \quad \left(1 \le m \le \frac{q}{2}\right). \tag{1.8}$$

Its transient eigenvalues are

$$\mu_{z,k} = 2\cos\frac{(2k+1)\pi}{2^{z+1}}, \qquad 0 \le z \le r-1, \quad 0 \le k < 2^z. \tag{1.9}$$

Equivalently, the transient spectrum is $\{2\cos(j\pi/q) : 1 \le j \le q-1\}$.

The projective vertices form a closed strongly connected component of period 1, and 2 is the unique eigenvalue of modulus 2. For $r \ge 2$, the next two distinct moduli are

$$2 > 2\cos\frac{\pi}{q+1} > 2\cos\frac{\pi}{q}. \tag{1.10}$$

The second is attained only by

$$\lambda^{\mathrm{sec}} = -2\cos\frac{\pi}{q+1}, \tag{1.11}$$

a projective eigenvalue. The third is attained precisely by the two transient eigenvalues $\pm 2\cos(\pi/q)$. Every remaining eigenvalue has smaller modulus. For $r = 1$, the spectrum is $\{2, -1, 0\}$.

For a 2-regular element $g \in \Gamma$, write

$$S(g) = \sum_{S \subseteq R} \chi_{L(S)}(g). \tag{1.12}$$

All these character values are real. Define also

$$S_q = \prod_{j=0}^{r-1} \left(1 + 2\cos\frac{2^{j+1}\pi}{q+1}\right). \tag{1.13}$$

Theorem 1.3 (The first correction)**.** *For every $r \ge 1$ and $n \ge 0$, there is an exact expansion*

$$b(n) = \sum_{[g]\ 2\text{-regular}} \frac{S(g)}{|C_\Gamma(g)|}\chi_V(g)^n + \sum_{z=0}^{r-1}\sum_{k=0}^{2^z-1} c_{z,k}\mu_{z,k}^n. \tag{1.14}$$

The constants $c_{z,k}$ are given explicitly by Proposition 7.1; for $n = 0$ we use $0^0 = 1$.

If $r \ge 2$, then S_q is a negative algebraic integer of degree $\varphi(q+1)/(2r)$ over $\mathbb{Q}$, and

$$b(n) = a(n) + \frac{S_q}{q+1}\left(-2\cos\frac{\pi}{q+1}\right)^n + \varepsilon(n), \qquad |\varepsilon(n)| \le K_q\left(2\cos\frac{\pi}{q}\right)^n \tag{1.15}$$

for a constant K_q and every $n \ge 0$. Both eigenvalues $\pm 2\cos(\pi/q)$ have positive coefficients in (1.14). In particular,

$$\lim_{n\to\infty}\left|\frac{b(n)}{a(n)} - 1\right|^{1/n} = \cos\frac{\pi}{q+1}. \tag{1.16}$$

For all sufficiently large n, one has $b(n) > a(n)$ when n is odd and $b(n) < a(n)$ when n is even.

If $r = 1$, then $S_2 = 0$ and $b(n) = a(n) = 2^{n-1}$ for every $n \ge 1$.

The projective spectrum thus governs the first correction, but only after comparison with the entire transient spectrum. The separation is small: Taylor expansion gives

$$\cos\frac{\pi}{q+1}-\cos\frac{\pi}{q}\sim\frac{\pi^2}{q^3}\qquad(q\to\infty). \tag{1.17}$$

The extreme transient eigenvalues also occur in the count itself, not just in the action matrix (their coefficients are given in Proposition 7.2).

1.2. **Outline of the argument.** We begin with Donkin's tensor product theorem, which describes $T(m)$ by digits in $\{1,2\}$. The identity

$$V^{\otimes 3}\cong(V\otimes V^{[1]})\oplus V^{\oplus 2}$$

turns tensoring by V into a carry operation on these digits. After restriction to Γ, the twists have period r, so the carry becomes cyclic: a carry from the last digit returns to the first.

The main difficulty is to pass from this finite family, which is closed under tensoring by V, to the indecomposable modules in the fusion graph. A state may still split on restriction. For states with a zero digit, we show that a weight bound preserves the endomorphism ring and hence indecomposability. For states without zeros, we instead factor off the Steinberg module and identify the projective summands by a dimension argument (only the state with every digit equal to 2 splits).

Once the indecomposable modules are identified, a quadratic recursion for the transient blocks produces the Chebyshev polynomials in (1.7), while Brauer character duality gives the projective spectrum and its coefficients. We then need two facts not supplied by diagonalization alone: the exact Galois stabilizer of S_q, which we find by folding a doubling orbit, and the extreme transient coefficients, which we evaluate by polynomial interpolation. These formulas give the examples in Section 8; the accompanying code provides supplementary calculations.

2. Tilting modules and character values

2.1. **Weights and Frobenius twists.** Let F be the endomorphism of G obtained by squaring matrix entries. Then $\Gamma=G^{F^r}$, and $X^{[j]}$ denotes pullback along F^j. We use the torus and root subgroups

$$T=\{\mathrm{diag}(a,a^{-1}):a\in k^\times\},\qquad U=\left\{u(t)=\begin{pmatrix}1&t\\0&1\end{pmatrix}:t\in k\right\},$$

and let U^- be the opposite root subgroup. Their $\mathbb{F}_q$-points are denoted by T_q, U_q, and U_q^-. The weight $\mu\in\mathbb{Z}$ is the character $\mathrm{diag}(a,a^{-1})\mapsto a^\mu$. In particular, $V=L(1)=\Delta(1)=\nabla(1)=T(1)$ has weights 1 and -1.

For a rational G-module X, the divided powers give

$$u(t)v=\sum_{i\ge0}t^ie^{(i)}v,\qquad e^{(i)}X_\mu\subseteq X_{\mu+2i}; \tag{2.1}$$

the corresponding formula for U^- uses $f^{(i)}$. We use the standard highest-weight and good-filtration properties of [5, Part II, Sections 1–4]. A U-fixed vector of weight μ defines a map from the Weyl module $\Delta(\mu)$ (and a nonzero such vector cannot have negative weight). With $B^+=TU$ and the convention $\Delta(\mu)=0$ for $\mu<0$, this gives

$$\mathrm{Hom}_{B^+}(k_\mu,X)\cong\mathrm{Hom}_G(\Delta(\mu),X)\qquad(\mu\in\mathbb{Z}).$$

If X has a good filtration, then for $\mu\ge0$,

$$\dim(X^U)_\mu=\dim\mathrm{Hom}_G(\Delta(\mu),X)=(X:\nabla(\mu)), \tag{2.2}$$

where $\nabla(\mu)$ is the induced module of highest weight μ.

2.2. **Simple modules and the two tori.** By Steinberg's tensor product theorem, the simple modules are the $L(S)$ introduced above, with

$$\dim L(S) = 2^{|S|}, \qquad L(\varnothing) = k, \qquad L(R) = \mathrm{St},$$

where the Steinberg module St is simple and projective of dimension q.

Choose generators a and b of a split and a nonsplit torus, respectively. Besides the identity, the 2-regular classes are represented by a^l for $1 \le l \le (q-2)/2$ and by b^m for $1 \le m \le q/2$. Their centralizers and character values are

g	$\lvert C_\Gamma(g)\rvert$	$\lvert g^\Gamma\rvert$	$\chi_V(g)$
1	$q(q^2-1)$	1	2
a^l	$q-1$	$q(q+1)$	$2\cos(2\pi l/(q-1))$
b^m	$q+1$	$q(q-1)$	$2\cos(2\pi m/(q+1))$.

(2.3)

We use here the usual defining-characteristic parameters for $\mathrm{SL}_2(q)$ from [4, Section 4A]. All the classes in (2.3) are real, and Frobenius squares the torus eigenvalues. Hence

$$\chi_{L(S)}(g) = \prod_{j \in S} \chi_V(F^j g). \tag{2.4}$$

Summing over the subsets gives, whenever $\chi_V(g) = 2\cos\theta$,

$$S(g) = \prod_{j=0}^{r-1} (1 + \chi_V(F^j g)) = \prod_{j=0}^{r-1} (1 + 2\cos(2^j\theta)). \tag{2.5}$$

At the identity this gives $S(1) = 3^r$.

2.3. **Digits for tilting modules.** Set

$$X_0 = k, \qquad X_1 = V, \qquad X_2 = V \otimes V.$$

The module $X_2 = T(2)$ is uniserial with successive composition factors $k, V^{[1]}, k$, and $T(3) = V \otimes V^{[1]}$ is simple. Tensor products of tilting modules are tilting, and their formal characters determine their decompositions. In rank one, Donkin's tensor product theorem [2, Proposition 2.1] gives

$$T(1+2b) \cong V \otimes T(b)^{[1]}, \qquad T(2+2b) \cong X_2 \otimes T(b)^{[1]} \qquad (b \ge 0). \tag{2.6}$$

See also [3] for the characteristic-2 tilting structure.

To apply (2.6) repeatedly, write a positive integer m in the form

$$m = \sum_{j=0}^{z-1} c_j 2^j, \qquad c_j \in \{1, 2\}. \tag{2.7}$$

We obtain the unique expansion by taking c_0 to be 1 or 2 according as m is odd or even, then repeating with $(m - c_0)/2$. One checks that the strings of length z parameterize exactly the interval $[2^z - 1, 2^{z+1} - 2]$. Iterating the tensor product theorem now gives

$$T(m) \cong \bigotimes_{j=0}^{z-1} X_{c_j}^{[j]}, \qquad \dim T(m) = 2^{\sum_j c_j}, \tag{2.8}$$

with the empty string representing $T(0) = k$.

Lemma 2.1. *As G-modules,*

$$V^{\otimes 3} \cong T(3) \oplus V^{\oplus 2} \cong (V \otimes V^{[1]}) \oplus V^{\oplus 2}. \tag{2.9}$$

Proof. We have the elementary character identity

$$(t+t^{-1})^3 = (t+t^{-1})(t^2+t^{-2}) + 2(t+t^{-1}),$$

whose right-hand side is the sum of the characters of $T(3)$ and two copies of V. Since all these modules, including $V^{\otimes 3}$, are tilting, the identity determines the decomposition. □

3. A cyclic tensor-product rule

For $c = (c_0, \dots, c_{r-1}) \in \{0,1,2\}^r$, define

$$M(c) = \bigotimes_{j=0}^{r-1} X_{c_j}^{[j]}\Big|_\Gamma. \tag{3.1}$$

We regard these modules as states, without yet assuming indecomposability. Tensoring at a digit equal to 0 or 1 increases that digit. At a digit equal to 2, the cube decomposition in Lemma 2.1 leaves a digit equal to 1 and either stops with multiplicity 2 or passes a copy of V to the next twist. The carry returns to digit 0 after the last twist, since F^r is the identity on Γ.

Write $[c]$ for a basis vector in the free abelian group on the states, and let $c^{(j)}$ be obtained from c by replacing its jth entry by 1. For $j \in R$, define

$$\mathrm{add}_j[c] = \begin{cases} [c+e_j], & c_j \le 1, \\ 2[c^{(j)}] + \mathrm{add}_{j+1}[c^{(j)}], & c_j = 2, \end{cases} \qquad \mathcal{A}[c] = \mathrm{add}_0[c]. \tag{3.2}$$

This recursion terminates even when the carry makes a complete turn: on returning to the starting site, it encounters a digit already changed to 1.

Proposition 3.1 (Tensor rule)**.** *For every $c \in \{0,1,2\}^r$ and $j \in R$, the expansion*

$$\mathrm{add}_j[c] = \sum_d a_d[d]$$

gives an isomorphism

$$V^{[j]} \otimes M(c) \cong \bigoplus_d M(d)^{\oplus a_d}. \tag{3.3}$$

In particular, $\mathcal{A}$ computes tensoring by V.

Proof. If $c_j \le 1$, the assertion is the identity $V^{[j]} \otimes X_{c_j}^{[j]} = X_{c_j+1}^{[j]}$. If $c_j = 2$, the jth twist of (2.9) gives

$$V^{[j]} \otimes M(c) \cong M(c^{(j)})^{\oplus 2} \oplus \big(V^{[j+1]} \otimes M(c^{(j)})\big).$$

We repeat on the second summand, following (3.2). At most one complete turn is needed, and the step from $r-1$ to 0 is valid because $F^r(g) = g$ for $g \in \Gamma$ identifies the two twists. We thus obtain (3.3). □

Corollary 3.2. *The states reachable from $(0,\dots,0)$ are exactly*

$$\mathcal{R} = \bigsqcup_{z=0}^{r} \{(c_0,\dots,c_{z-1},0,\dots,0) : c_j \in \{1,2\}\}. \tag{3.4}$$

There are $2q-1$ such states. Every indecomposable summand of a tensor power of V occurs in some $M(c)$ with $c \in \mathcal{R}$, and every such state module occurs as a direct summand of a tensor power. The states with no zero entries are closed under $\mathcal{A}$.

Proof. A carry creates a new nonzero digit only at the first zero it reaches, and never makes a nonzero digit zero. Starting from the zero state, the nonzero digits thus remain an initial segment, which gives one inclusion in (3.4). The states without zeros remain closed under the rule.

For the reverse inclusion, we label the states by the integer m in (2.7), obtaining the labels $0, 1, \dots, 2q-2$. When $m < 2q-2$, following the carry branch until it stops changes the label from m to $m+1$ (there is no wraparound before the all-2 state). Thus every displayed state is reachable from label 0, and the tensor rule of Proposition 3.1 gives the assertions about direct summands. □

4. Restriction and the transient blocks

By (2.8), the reachable states with a zero digit are precisely $T(m)|_\Gamma$ for $0 \le m \le q-2$, since their digit length is at most $r-1$. We need to know that restriction does not introduce further decompositions. For this, we show that a weight bound preserves the entire endomorphism ring.

Lemma 4.1 (Restriction of homomorphisms)**.** *Let X and Y be finite-dimensional rational G-modules. Suppose that every weight μ of $X^* \otimes Y$ is even and satisfies $|\mu| \le 2q-4$. Then restriction induces*

$$\mathrm{Hom}_G(X,Y) = \mathrm{Hom}_\Gamma(X,Y). \tag{4.1}$$

Proof. Put $W = X^* \otimes Y$, and let $v \in W^\Gamma$. As v is fixed by T_q, its weight components have weights divisible by $q-1$. These weights are even and satisfy $|\mu| < 2(q-1)$, whereas $q-1$ is odd, so $v \in W_0$.

For $t \in \mathbb{F}_q$, the divided-power formula (2.1) gives

$$0 = u(t)v - v = \sum_{i=1}^{q-2} t^i e^{(i)} v.$$

Here the weight bound removes all terms with larger i. Each coordinate is a polynomial of degree at most $q-2$ vanishing on $\mathbb{F}_q^\times$, hence is zero. We obtain $e^{(i)}v = 0$ for every $i \ge 1$, so v is fixed by U. Repeating with U^- shows that v is fixed by $G = \langle U, U^- \rangle$, giving $W^\Gamma = W^G$ and hence (4.1). □

Proposition 4.2. *The modules $T(m)|_\Gamma$, for $0 \le m \le q-2$, are indecomposable, pairwise nonisomorphic, and nonprojective.*

Proof. We apply Lemma 4.1 with both modules equal to $T(m)$. This module is self-dual and all its weights have the parity of m, so the weights of $T(m)^* \otimes T(m)$ are even, with absolute value at most $2m \le 2q-4$. Thus

$$\mathrm{End}_\Gamma(T(m)) = \mathrm{End}_G(T(m)),$$

and this ring is local because $T(m)$ is indecomposable. The restriction is therefore indecomposable as well.

We distinguish the restrictions by recovering m from the U_q-fixed vectors. For every $v \in T(m)$, the polynomial $u(t)v - v$ has degree at most m, since the weights of $T(m)$ lie in $[-m, m]$. The same polynomial argument applies, as $m \le q-2$, and gives

$$T(m)^{U_q} = T(m)^U.$$

The good-filtration identity (2.2) places the weights of this space in $[0, m]$, with multiplicities given by those in a good filtration of $T(m)$. The largest weight is m. Since the integers in $[0, q-2]$ give distinct characters of T_q, we can recover these weights, and in particular m, from the T_q-module $T(m)^{U_q}$.

Let n_1 and n_2 count the digits 1 and 2 in (2.7), with sum at most $r-1$. If $T(m)|_\Gamma$ were projective, its restriction to the Sylow 2-subgroup U_q would be free, giving

$$\dim T(m)^{U_q} = \frac{\dim T(m)}{q} = 2^{n_1+2n_2-r} \le 2^{n_2-1}. \tag{4.2}$$

On the other hand, V^U contains a line, and X_2^U contains the two independent vectors

$$e_1 \otimes e_1, \qquad e_1 \otimes e_2 + e_2 \otimes e_1.$$

Tensoring these fixed spaces, with their Frobenius twists, gives

$$\dim T(m)^{U_q} = \dim T(m)^U \geq 2^{n_2},$$

contradicting (4.2). □

4.1. **The quadratic recursion.** Before the carry reaches digit $r-1$, we are still applying identities of rational G-modules. A state of length $z < r$ either stays at length z or passes to length $z+1$; it cannot shorten. We thus obtain a block lower-triangular transient matrix by grouping states according to length.

Proposition 4.3. *Let B_z be the part of the tensor rule that stays among digit strings of length z, with a carry out of that length discarded. Then $B_0 = (0)$. After grouping strings by their first digit,*

$$B_{z+1} = \begin{pmatrix} 0 & 2I + B_z \\ I & 0 \end{pmatrix} = N \otimes I_{2^z} + E \otimes B_z, \qquad N = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{4.3}$$

For $0 \leq z \leq r-1$, these are the diagonal blocks of B. Its only possible off-diagonal transitions go from length z to length $z+1$.

If $p_z(x) = \det(xI - B_z)$, then

$$p_0(x) = x, \qquad p_{z+1}(x) = p_z(x^2 - 2) = p_z(x)^2 - 2 = 2\mathsf{T}_{2^{z+1}}(x/2). \tag{4.4}$$

In particular, the roots of p_z are the 2^z distinct numbers $\mu_{z,k}$ in (1.9).

Proof. We group the strings by their first digit. A digit equal to 1 changes to 2 without affecting the remaining digits, while a digit equal to 2 changes to 1 and contributes either the stop term $2I$ or the carry term B_z. These are the two columns of (4.3). Since a carry creates at most one new digit, the off-diagonal transitions have the stated form.

The block determinant in (4.3) gives

$$\det(xI - B_{z+1}) = \det((x^2 - 2)I - B_z) = p_z(x^2 - 2).$$

Starting with $p_0 = x = 2\mathsf{T}_1(x/2)$, we obtain (4.4) from the standard Chebyshev composition and doubling identities. In particular,

$$p_z(2\cos\theta) = 2\cos(2^z\theta),$$

which yields the stated roots and their simplicity. □

For comparison with the usual tilting tensor rule, (2.6) and (2.9) give

$$\begin{aligned} V \otimes T(m) &\cong T(m+1) && \text{if } m \text{ is odd,} \\ V \otimes T(2+2b) &\cong T(1+2b)^{\oplus 2} \oplus \Psi(V \otimes T(b)), \end{aligned} \tag{4.5}$$

where Ψ is additive on tilting modules and sends $T(a)$ to $T(1+2a)$. The initial rule is $V \otimes T(0) = T(1)$. For example,

$$V \otimes T(6) \cong T(7) \oplus T(5)^{\oplus 2} \oplus T(3)^{\oplus 2}.$$

These are the characteristic-2 tilting rules used in the algebraic-group tensor-growth problem [8, Section 2] (we do not need the algebraic-group asymptotic estimates).

5. The projective modules and their spectrum

5.1. **Decomposing the full-length states.** For $c \in \{1,2\}^r$, the digit formula gives

$$M(c) = \mathrm{St} \otimes L(B_c), \tag{5.1}$$

where $B_c = \{j : c_j = 2\}$. These full-length states are therefore projective. We identify their summands by locating one projective cover in each state and comparing the resulting dimension bounds with the dimension of the regular module.

Proposition 5.1. *For $B \subsetneq R$, one has*

$$\mathrm{St} \otimes L(B) \cong P(R \setminus B), \qquad \dim P(R \setminus B) = q2^{|B|}. \tag{5.2}$$

The remaining state satisfies

$$\mathrm{St} \otimes \mathrm{St} \cong P(\varnothing) \oplus \mathrm{St}, \qquad \dim P(\varnothing) = q(q-1). \tag{5.3}$$

Every projective indecomposable is therefore a vertex of the fusion graph. Among the reachable state modules, only the all-2 state is decomposable, and it has two indecomposable summands.

Proof. All simple modules are self-dual, and St is simple and projective. Thus, for $T \subseteq R$, tensor–Hom adjunction gives

$$\begin{aligned} [\mathrm{St} \otimes L(B) : P(T)]_\oplus &= \dim \mathrm{Hom}_\Gamma(\mathrm{St} \otimes L(B), L(T)) \\ &= \dim \mathrm{Hom}_\Gamma(\mathrm{St}, L(B) \otimes L(T)) \\ &= [L(B) \otimes L(T) : \mathrm{St}]. \end{aligned} \tag{5.4}$$

We take $T = R \setminus B$, so the module in the last line is St. The cover $P(R \setminus B)$ occurs once, giving the dimension bounds

$$\dim P(T) \le q2^{r-|T|} \qquad (T \subseteq R). \tag{5.5}$$

For $B = R$, we need to account for the Steinberg summand of $\mathrm{St}^{\otimes 2}$ as well. The Steinberg character takes the values q, 1, and -1 at the identity, the nonidentity split semisimple elements, and the nonsplit semisimple elements, respectively. Using the class data (2.3), the projective character pairing gives

$$\begin{aligned} [\mathrm{St}^{\otimes 2} : \mathrm{St}]_\oplus &= \frac{q^3 + \frac{q-2}{2}q(q+1) - \frac{q}{2}q(q-1)}{q(q^2-1)} \\ &= 1. \end{aligned} \tag{5.6}$$

We can thus remove a Steinberg summand in addition to $P(\varnothing)$ from $\mathrm{St}^{\otimes 2}$, improving the bound for the trivial cover to $\dim P(\varnothing) \le q^2 - q$.

We now compare with the regular-module identity

$$\sum_{T \subseteq R} 2^{|T|} \dim P(T) = |\Gamma| = q^3 - q. \tag{5.7}$$

The upper bounds (5.5) give q^3 in this sum before the improvement at $T = \varnothing$, and $q^3 - q$ after it. Thus equality must hold in every bound, leaving no further summands in either (5.2) or (5.3). By Corollary 3.2, each of these projectives occurs in a tensor power. □

To pass from the state matrix to the action matrix on indecomposables, we need only change the last basis vector. Label the state modules by $m = 0, \dots, 2q-2$ as in (2.7), and replace the last state by $P(\varnothing)$ through

$$[M(2,\dots,2)] = [P(\varnothing)] + [\mathrm{St}], \qquad \mathrm{St} = M(1,\dots,1) = T(q-1)|_\Gamma. \tag{5.8}$$

All the other states already represent vertices, so this change of basis specifies C and $\widetilde{M}$ in (1.6).

Proof of Theorem 1.1. By Corollary 3.2, the vertices are exactly the indecomposable summands of the reachable state modules. The $q-1$ states of length less than r are the nonprojective indecomposables of Proposition 4.2, while the full-length states give exactly the q projective indecomposables of Proposition 5.1. Each occurs in a tensor power.

The upper-right block of M is zero because tensoring preserves projectivity. Before the carry reaches full length, the tensor identities hold for G itself, so B is the truncated algebraic-group action matrix, with the diagonal blocks and characteristic polynomials of Proposition 4.3. □

We can now compare the finite-group and algebraic-group counts exactly. For $m \geq q-1$, the tilting digit expansion has length at least r, so the first r factors in (2.8) contain St on restriction. Thus $T(m)|_\Gamma$ is projective, and for every n the number of nonprojective summands of $V^{\otimes n}|_\Gamma$ equals

$$\sum_{m=0}^{q-2} [V^{\otimes n} : T(m)]_{\oplus,G}. \tag{5.9}$$

For $m \leq 2q-3$, the restriction of $T(m)$ is indecomposable by the preceding propositions, and no tilting summand of $V^{\otimes n}$ has highest weight greater than n. Hence the two counts agree initially:

$$b^{\Gamma,V}(n) = b^{G,V}(n) \qquad (0 \leq n \leq 2q-3). \tag{5.10}$$

5.2. **Character diagonalization.** We give the projective calculation for a general finite group H, retaining complex conjugation. Let W be a kH-module in characteristic $p \geq 0$. Write $g_1, \ldots, g_N$ for the p-regular class representatives, and φ_i, Φ_i for the characters of the simple modules L_i and their projective covers P_i (ordinary characters in characteristic zero).

Proposition 5.2. *On the span of all projective indecomposables, the matrix*

$$\widetilde{M}_{ji} = [W \otimes P_i : P_j]_\oplus$$

is diagonalizable. For each p-regular g, the vector

$$v_g = (\overline{\varphi_i(g)})_i \tag{5.11}$$

is a right eigenvector of eigenvalue $\chi_W(g)$. Equivalently, $(\varphi_i(g))_i$ is a right eigenvector of eigenvalue $\chi_{W^*}(g)$, and these two descriptions give the same multiset of eigenvalues.

For the full fusion graph, character evaluation $w_g = (\chi_X(g))_X$ is a left eigenvector of eigenvalue $\chi_W(g)$. If all P_i are vertices, extend v_g by zero on the nonprojective vertices. Then

$$w_g^{\mathsf{T}} v_h = |C_H(g)|\, \mathbf{1}_{g\sim h} \tag{5.12}$$

for p-regular g, h.

Proof. Tensor–Hom adjunction and projectivity give

$$\widetilde{M}_{ji} = \dim \mathrm{Hom}_H(W \otimes P_i, L_j) = \dim \mathrm{Hom}_H(P_i, W^* \otimes L_j) = [W^* \otimes L_j : L_i].$$

Evaluating characters, we see that $(\varphi_i(g))_i$ has eigenvalue $\chi_{W^*}(g)$, and complex conjugation gives (5.11). These vectors form a basis because the Brauer character table is nonsingular. The two eigenvalue multisets agree because inversion permutes the p-regular classes.

For the left eigenvector, we evaluate the column indexed by X:

$$(w_g^{\mathsf{T}} M)_X = \sum_Y \chi_Y(g)[W \otimes X : Y]_\oplus = \chi_W(g)\chi_X(g).$$

The standard Brauer duality relation [9, Chapter 2] is

$$\sum_i \Phi_i(g)\overline{\varphi_i(h)} = |C_H(g)|\, \mathbf{1}_{g\sim h},$$

which gives (5.12) (the eigenvector identities also appear in [4, Lemma 7]). □

For an arbitrary W, one need not find all projective indecomposables in its fusion graph. Proposition 5.2 concerns the full projective span, and we may apply it here because every $P(S)$ occurs by Proposition 5.1. In particular, any eigenvalue of a finite fusion graph that is not a Brauer character value must come from the nonprojective block.

Corollary 5.3. *For the natural module of Γ, the projective block has the q distinct eigenvalues in* (1.8). *It is irreducible of period* 1. *The eigenvalue* 2 *is simple and is its only eigenvalue of modulus* 2. *The largest remaining modulus is $2\cos(\pi/(q+1))$, attained only at $\chi_V(b^{q/2}) = -2\cos(\pi/(q+1))$.*

Proof. We obtain the spectrum from (2.3) and Proposition 5.2. The angles in each torus lie strictly between 0 and π, so their cosines are distinct. Equality between a split and a nonsplit value would give

$$\frac{l}{q-1} = \frac{m}{q+1}.$$

As $\gcd(q-1, q+1) = 1$, this would force $q-1$ to divide l, contrary to its range. None of the nonidentity values is 2 or -2.

For irreducibility, we use the projective dimensions as a strictly positive left eigenvector for 2, and the simple dimensions as a strictly positive right eigenvector. Conjugating $\widetilde{M}/2$ by the diagonal matrix of projective dimensions gives a stochastic matrix with a strictly positive stationary vector. By the standard finite-state argument, all communicating classes are closed, and simplicity of 2 forces a single class. The period is 1, since no other eigenvalue has modulus 2.

For $q \geq 4$, the largest split modulus is $2\cos(\pi/(q-1))$, whereas the largest nonsplit modulus is $2\cos(\pi/(q+1))$. The latter is larger and is attained only at $m = q/2$. For $q = 2$, the conclusion follows from the projective spectrum $\{2, -1\}$ (the split family is empty). □

Proof of Theorem 1.2. The block decomposition (1.6) reduces the spectrum to those of B and $\widetilde{M}$, which are given by Proposition 4.3 and Corollary 5.3. We need to check that the two families are disjoint and compare their moduli.

The roots of p_z form the Galois orbit of $2\cos(\pi/2^{z+1})$, of degree 2^z. They are simple, and different values of z are distinguished by the reduced angular denominators. These roots lie in $\mathbb{Q}(\zeta_{2^{z+2}})$, while the projective eigenvalues lie in $\mathbb{Q}(\zeta_{q^2-1})$. By the standard intersection property for cyclotomic fields of coprime conductors, their intersection is $\mathbb{Q}$. The only rational transient eigenvalue is 0, and this is not projective (a root of unity satisfying $z + z^{-1} = 0$ has order 4, not odd order). Thus all $2q-1$ eigenvalues are distinct, and M is diagonalizable.

To obtain the other description of the transient spectrum, we write each $1 \leq j < q$ as a power of 2 times an odd integer. This recovers exactly the angles in (1.9).

Suppose $r \geq 2$. The largest transient modulus is $\alpha_q = 2\cos(\pi/q)$, attained only by α_q and $-\alpha_q$ in B_{r-1}, with all remaining transient moduli at most $\max\{2\cos(3\pi/q), 2\cos(2\pi/q)\} < \alpha_q$. Among the nonperipheral projective eigenvalues, removing the extreme nonsplit value leaves moduli at most

$$\max\left\{2\cos\frac{\pi}{q-1},\, 2\cos\frac{2\pi}{q+1}\right\} < 2\cos\frac{\pi}{q}.$$

Since $2\cos(\pi/(q+1)) > 2\cos(\pi/q)$, we obtain (1.10) with the stated equality cases. For $r = 1$, we use $B_0 = (0)$ and the projective spectrum in the final sentence of Corollary 5.3. □

6. The first correction and its coefficient

For a simple eigenvalue λ of a finite matrix, we can recover its coefficient in (1.2) from right and left eigenvectors v_λ, w_λ by the usual spectral projection:

$$c_\lambda = \frac{(\mathbf{1}^{\mathsf{T}} v_\lambda)(w_\lambda^{\mathsf{T}} e_0)}{w_\lambda^{\mathsf{T}} v_\lambda}. \tag{6.1}$$

For the projective eigenvalue $\chi_V(g)$, character duality in Proposition 5.2 gives the pairing $|C_\Gamma(g)|$ and the right-vector sum $S(g)$. Since $w_g^{\mathsf{T}} e_0 = 1$ is the character value of the trivial module, the coefficient reduces to

$$c_{\chi_V(g)} = \frac{S(g)}{|C_\Gamma(g)|}, \tag{6.2}$$

which is the projective part of (1.14) (at the identity it gives the leading coefficient $3^r/[q(q^2-1)]$).

We apply this to the subdominant class $b^{q/2}$. Put $N = q+1$. The congruence $2^r \equiv -1 \pmod N$ shows that multiplication by 2 permutes the factors in

$$\prod_{j=0}^{r-1} \left(1 + 2\cos\frac{2\pi\, 2^j a}{N}\right).$$

Thus the exponent $q/2 = 2^{r-1}$ gives the same product as $a = 1$, and $S(b^{q/2}) = S_q$. For $r \geq 2$, this product has exactly one negative factor, the last, and all other factors exceed 1. More explicitly,

$$S_q = -\left(2\cos\frac{\pi}{q+1} - 1\right) \prod_{j=0}^{r-2} \left(1 + 2\cos\frac{2^{j+1}\pi}{q+1}\right) < 0. \tag{6.3}$$

The subdominant coefficient is therefore nonzero.

6.1. **The exact degree of S_q.** Doubling and inversion fix (1.13), giving an upper bound for its degree from the subgroup they generate in $\mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$. To prove equality, we must exclude any larger stabilizer. We do this by showing that the product on this doubling orbit has strictly larger absolute value than the products on every other orbit.

Lemma 6.1. *Let $r \geq 2$, let $N = 2^r + 1$, and put*

$$H = \{\pm 2^j \bmod N : 0 \leq j < r\}, \qquad F(a) = \prod_{j=0}^{r-1} \left(1 + 2\cos\frac{2\pi\, 2^j a}{N}\right).$$

Then $|H| = 2r$, and

$$|F(a)| \leq 3^{r-3} < |F(1)| \tag{6.4}$$

for every integer $1 \leq a < N$ whose residue is not in H.

Proof. The order of H is $2r$: we have $2^r \equiv -1 \pmod N$, while for $0 < j < r$ neither $2^j - 1$ nor $2^j + 1$ is divisible by N.

We fold the doubling orbit by taking t_j to be the least absolute residue of $2^j a$ modulo N, in $\{1, \ldots, (N-1)/2\}$. Then $t_r = t_0$, and the folded orbit obeys

$$t_{j+1} = \begin{cases} 2t_j, & t_j < N/4, \\ N - 2t_j, & t_j > N/4. \end{cases} \tag{6.5}$$

There is no equality case at $N/4$, since N is odd. Let ν count the indices at which the second branch is taken. Reducing the iteration modulo N, we obtain

$$t_0 \equiv (-1)^\nu 2^r t_0 \equiv (-1)^{\nu+1} t_0 \pmod N.$$

Since $0 < 2t_0 < N$, the number ν is odd.

If $\nu = 1$, we rotate the periodic sequence to place its unique second-branch step last. All preceding steps are doublings, so closure of the cycle gives

$$t_0 = N - 2^r t_0.$$

Hence $t_0 = 1$ and the original residue a lies in H. For $a \notin H$, we must therefore have $\nu \geq 3$. At each second-branch step the angle $2\pi t_j/N$ lies between $\pi/2$ and π, giving

$$\left|1 + 2\cos\frac{2\pi t_j}{N}\right| \leq 1.$$

All other factors have absolute value at most 3, giving the first inequality in (6.4).

For the second inequality, put $\theta_j = 2^{j+1}\pi/N$. The penultimate factor of $F(1)$ exceeds 1, and the absolute value of the last is $2\cos(\pi/N) - 1 > 1/2$. We bound the preceding factors by $\cos t \geq 1 - t^2/2$, obtaining

$$\prod_{j=0}^{r-3} \frac{1 + 2\cos\theta_j}{3} \geq \prod_{j=0}^{r-3} \left(1 - \frac{\theta_j^2}{3}\right)$$

(the product is empty when $r = 2$). The geometric sum is

$$\sum_{j=0}^{r-3} \theta_j^2 = \frac{4\pi^2}{N^2}\,\frac{4^{r-2} - 1}{3} < \frac{\pi^2}{12},$$

so the elementary inequality $\prod_j (1 - u_j) \geq 1 - \sum_j u_j$ for $0 \leq u_j \leq 1$ gives

$$\prod_{j=0}^{r-3} \left(1 - \frac{\theta_j^2}{3}\right) \geq 1 - \frac{1}{3}\sum_{j=0}^{r-3} \theta_j^2 > 1 - \frac{\pi^2}{36} > \frac{2}{3}.$$

Combining the three groups of factors, we obtain $|F(1)| > 3^{r-2}(2/3)(1/2) = 3^{r-3}$. □

Corollary 6.2. *For $r \geq 2$, the algebraic integer S_q has degree $\varphi(q+1)/(2r)$ over $\mathbb{Q}$.*

Proof. Put $\zeta = \zeta_{q+1}$. The factors $1 + \zeta^{2^j} + \zeta^{-2^j}$ are algebraic integers, and $\sigma_a(\zeta) = \zeta^a$ sends their product to $F(a)$. By Lemma 6.1, its stabilizer in $(\mathbb{Z}/(q+1)\mathbb{Z})^\times$ is exactly H: all elements of H fix $F(1)$, and the other images have smaller absolute value. Thus the orbit has size $\varphi(q+1)/|H| = \varphi(q+1)/(2r)$. □

Proof of the expansion and asymptotics in Theorem 1.3. We obtain (1.14) by applying the spectral projection formula (6.1) to Theorem 1.2. The projective coefficients are given by (6.2), with subdominant coefficient $S_q/(q+1)$ from the class calculation above. This coefficient is negative by (6.3), while the integrality and degree assertion for S_q is Corollary 6.2.

For $r \geq 2$, all eigenvalues other than 2 and λ^{sec} have modulus at most $2\cos(\pi/q)$. We may take K_q in (1.15) to be the sum of the absolute values of their coefficients. Since $|\lambda^{\mathrm{sec}}| > 2\cos(\pi/q)$ and the subdominant coefficient is nonzero, dividing by (1.5) gives (1.16) and the claimed parity of the error.

For $r = 1$, we have $S_2 = 1 + 2\cos(2\pi/3) = 0$ and the only transient eigenvalue is 0, so $b(n) = 2^{n-1}$ for $n \geq 1$. One can also read this directly from the tensor rules

$$V \otimes V \cong P(\varnothing) \oplus V, \qquad V \otimes P(\varnothing) \cong V^{\oplus 2}.$$

The remaining coefficient assertions are proved in Section 7. □

7. The transient coefficients

We can recover the transient coefficients directly from the triangular matrix. Let μ be an eigenvalue of the full transient matrix B, with right and left eigenvectors u and ℓ. Since μ is absent from the projective spectrum, we can extend these to eigenvectors of M by

$$v_\mu = \begin{pmatrix} u \\ (\mu I - \widetilde{M})^{-1} C u \end{pmatrix}, \qquad w_\mu^{\mathsf{T}} = (\ell^{\mathsf{T}}, 0). \tag{7.1}$$

The projection formula (6.1) then gives

$$c_\mu = \frac{\big(\mathbf{1}^\mathsf{T}u + \mathbf{1}^\mathsf{T}(\mu I - \widetilde{M})^{-1}Cu\big)(\ell^\mathsf{T}e_0)}{\ell^\mathsf{T}u}. \tag{7.2}$$

For the diagonal blocks, one can construct the eigenvectors recursively: if $B_z u = \nu u$ and $\mu^2 = 2+\nu$, then

$$B_{z+1}\begin{pmatrix}\mu u\\ u\end{pmatrix} = \mu\begin{pmatrix}\mu u\\ u\end{pmatrix}.$$

We then solve through the remaining triangular blocks to obtain the eigenvectors of B (the block spectra are disjoint).

The resolvent formula does not make the signs of the extreme coefficients apparent. To evaluate these, we instead realize the count as a linear functional on a polynomial quotient, obtaining a scalar formula for every coefficient in (1.14).

7.1. **Polynomial interpolation.** We continue the recursion $p_0(x) = x$, $p_{j+1}(x) = p_j(x)^2 - 2$ through $j = r$. For $0 \le m \le 2q-2$, the tilting digits of m give the polynomial

$$t_m(x) = \prod_{j=0}^{r-1} p_j(x)^{c_j}, \tag{7.3}$$

where we pad the digits with zeros to length r. This polynomial is monic of degree m. By $p_j(t + t^{-1}) = t^{2^j} + t^{-2^j}$ and the digit formula (2.8), we have

$$t_m([V]) = [T(m)] \tag{7.4}$$

in the split Grothendieck ring of tilting G-modules, since formal characters are injective on this ring. The identity remains valid after restriction to Γ.

We claim that the characteristic polynomial of M is

$$D_r(x) = \left(\prod_{j=0}^{r-1} p_j(x)\right)(p_r(x) - x). \tag{7.5}$$

Indeed, the first factor supplies the transient spectrum. For $x = 2\cos\theta$, the second vanishes precisely when $q\theta \equiv \theta$ or $q\theta \equiv -\theta \pmod{2\pi}$. The two tori in (1.8) give q distinct roots, exhausting its degree, so the roots of $p_r(x) - x$ are exactly the projective eigenvalues.

The polynomials $t_0, \dots, t_{2q-2}$ form a basis of the polynomials of degree at most $2q-2$. To count summands, we assign the value 1 to each basis polynomial except the last, which receives 2 because of (5.3). Thus we define the linear functional $\mathcal{L}_r$ on this space by

$$\mathcal{L}_r(t_m) = \begin{cases} 1, & 0 \le m \le 2q-3, \\ 2, & m = 2q-2. \end{cases} \tag{7.6}$$

Proposition 7.1. *For every $n \ge 0$,*

$$b(n) = \mathcal{L}_r(x^n \bmod D_r). \tag{7.7}$$

For every root λ of D_r, its coefficient in (1.14) *is*

$$c_\lambda = \frac{\mathcal{L}_r\big(D_r(x)/(x-\lambda)\big)}{D_r'(\lambda)}, \tag{7.8}$$

where $\mathcal{L}_r$ is extended to the field containing λ.

Proof. By the change of basis (5.8), the $2q-1$ state classes form a basis of the span of the graph vertices (only the final class differs from an indecomposable class, by the addition of the Steinberg class). Monicity of the t_m and (7.4) then show that $e_0, Me_0, \dots, M^{2q-2}e_0$ are also a basis. We may thus identify this span with $\mathbb{Q}[x]/(D_r)$ by $f \mapsto f(M)e_0$. Under this identification, the counting functional is (7.6), giving (7.7).

Since the roots of D_r are simple, Lagrange interpolation gives

$$x^n \equiv \sum_{D_r(\lambda)=0} \lambda^n \frac{D_r(x)}{(x-\lambda)D_r'(\lambda)} \pmod{D_r(x)},$$

and we obtain (7.8) by applying $\mathcal{L}_r$. □

7.2. **The two extreme transient terms.** For the roots of p_{r-1}, we can evaluate (7.8) explicitly. The quadratic recursion factors the divided difference into digit polynomials, so that the counting functional can be applied term by term.

Proposition 7.2. *Let $r \geq 2$. For each root μ of p_{r-1}, the coefficient of μ^n in (1.14) is*

$$c_\mu = \frac{(2-\mu)(1+\mu)^2}{2q} \prod_{j=1}^{r-2} (1 + p_j(\mu)). \tag{7.9}$$

In particular, put

$$\alpha_q = 2\cos\frac{\pi}{q}, \qquad R_q = \prod_{j=1}^{r-2} \left(1 + 2\cos\frac{2^j\pi}{q}\right).$$

With an empty product interpreted as 1, the coefficients of α_q^n and $(-\alpha_q)^n$ are

$$c_+ = \frac{(2-\alpha_q)(1+\alpha_q)^2}{2q} R_q, \qquad c_- = \frac{(2+\alpha_q)(1-\alpha_q)^2}{2q} R_q. \tag{7.10}$$

Both are strictly positive.

Proof. We first evaluate the numerator $\mathcal{L}_r(D_r/(x-\mu))$ in (7.8). Put $d = q/2$, $P = p_{r-1}$, and $A = \prod_{j=0}^{r-2} p_j$. To express the numerator in the tilting basis, we use $Q_\mu = P/(x-\mu)$, whose factorization is

$$Q_\mu(x) = \prod_{j=0}^{r-2} (p_j(x) + p_j(\mu)). \tag{7.11}$$

This is repeated difference-of-squares factorization: $P(\mu) = 0$, and each factorization of $p_{j+1}(x) - p_{j+1}(\mu)$ removes $p_j(x) - p_j(\mu)$ until only $x - \mu$ remains.

For $J \subseteq \{0, \dots, r-2\}$, write $h_J = \prod_{j \in J} p_j$, and expand

$$Q_\mu = \sum_J a_J h_J, \qquad s_\mu = \sum_J a_J = \prod_{j=0}^{r-2} (1 + p_j(\mu)).$$

The coefficient of $h_{\{0,\dots,r-2\}}$ is 1. Each Ah_J is transient, giving $\mathcal{L}_r(AQ_\mu) = s_\mu$, while each AP^2h_J is a full-length tilting polynomial with final digit 2. Only the full set J gives the all-2 polynomial, and it does so with coefficient 1. We therefore have

$$\mathcal{L}_r(AQ_\mu) = s_\mu, \qquad \mathcal{L}_r(AP^2Q_\mu) = s_\mu + 1, \qquad \mathcal{L}_r(AP) = 1. \tag{7.12}$$

Using $xQ_\mu = P + \mu Q_\mu$, we also obtain

$$\mathcal{L}_r(AxQ_\mu) = 1 + \mu s_\mu.$$

As $D_r = AP(P^2 - 2 - x)$, substitution gives

$$\mathcal{L}_r\left(\frac{D_r}{x-\mu}\right) = (s_\mu + 1) - 2s_\mu - (1 + \mu s_\mu) = -(1+\mu)s_\mu. \tag{7.13}$$

For the denominator, the same recursion gives $P' = dA$, and differentiation at the root μ yields

$$D_r'(\mu) = -(2+\mu)dA(\mu)^2.$$

Write $\mu = 2\cos\theta$ with $0 < \theta < \pi$. We have $\cos(d\theta) = 0$ because $P(\mu) = 0$, so the usual sine product gives

$$A(\mu) = \frac{\sin(d\theta)}{\sin\theta}, \qquad A(\mu)^2 = \frac{1}{\sin^2\theta}.$$

Substituting this and (7.13) into (7.8), and using $\sin^2\theta = (4-\mu^2)/4$, gives

$$c_\mu = \frac{(1+\mu)s_\mu \sin^2\theta}{(2+\mu)d} = \frac{(2-\mu)(1+\mu)^2}{2q}\prod_{j=1}^{r-2}(1 + p_j(\mu)).$$

For $\mu = \pm\alpha_q$, both values of $p_j(\mu)$ equal $2\cos(2^j\pi/q)$ when $j \geq 1$. Every factor in R_q is positive, and $1 < \alpha_q < 2$, so both coefficients in (7.10) are positive. □

Propositions 7.1 and 7.2 supply the remaining coefficient assertions in Theorem 1.3. In particular, after subtracting all projective terms, the error scale $2\cos(\pi/q)$ is attained on the even subsequence, where its leading coefficient is $c_+ + c_- > 0$.

8. Examples and exact coefficient calculations

8.1. **The fifteen-vertex graph for $\mathrm{SL}_2(8)$.** For $r = 3$, we order the vertices by the state labels $m = 0, \ldots, 14$, replacing the last state by $P(\varnothing)$ as in (5.8). In this order, the dimensions are

$$1, 2, 4, 4, 8, 8, 16, 8, 16, 16, 32, 16, 32, 32, 56.$$

The transient dimensions are thus $1, 2, 4, 4, 8, 8, 16$, while the projective dimensions form the multiset $8, 16, 16, 16, 32, 32, 32, 56$. Applying the carry rule in this basis gives the matrix of [4, Figure 9], with characteristic polynomial

$$\begin{aligned} D_3(x) = x(x-2)(x+1)(x^2-2)(x^3-3x+1) \\ \cdot (x^4 - 4x^2 + 2)(x^3 + x^2 - 2x - 1). \end{aligned} \tag{8.1}$$

Here x, x^2-2, and x^4-4x^2+2 come from the three transient blocks. The nonsplit and split tori contribute $(x+1)(x^3-3x+1)$ and x^3+x^2-2x-1, respectively (the remaining root is 2).

We can identify the subdominant eigenvalue in He's notation by setting $\omega = e^{2\pi i/18}$. The cyclotomic relations $\omega^8 = \omega^5 - \omega^2$ and $\omega^{-1} = -\omega^8$ give

$$\omega^5 - \omega^2 - \omega = -\omega^{-1} - \omega = -2\cos\frac{\pi}{9}.$$

Here $S_8 = -3$, and hence

$$b(n) = \frac{3}{56}\,2^n - \frac{1}{3}\left(-2\cos\frac{\pi}{9}\right)^n + O\left(\left(2\cos\frac{\pi}{8}\right)^n\right).$$

TABLE 1. Minimal polynomials of S_q. The degrees in the last two rows follow from Corollary 6.2; their coefficients are obtainable from (8.2)–(8.3).

r	q	Minimal polynomial or degree
1	2	X
2	4	$X+1$
3	8	$X+3$
4	16	X^2+8X-1
5	32	$X^2+22X-11$
6	64	$X^4+56X^3-514X^2+1096X+1$
7	128	Degree 6
8	256	Degree 16

8.2. **Minimal polynomials of the correction coefficient.** We can compute the minimal polynomial of S_q explicitly from its conjugates, whose number is given by Corollary 6.2. Put $N=q+1$, $d=\varphi(N)/(2r)$, and

$$f_N(X)=\prod_{j=0}^{r-1}(1+X^{2^j}+X^{N-2^j}) \pmod{\Phi_N(X)}.$$

The conjugates of S_q are indexed by $(\mathbb{Z}/N\mathbb{Z})^\times/H$. Rather than choose representatives, we compute their power sums by taking cyclotomic traces. Write

$$f_N(X)^k\equiv\sum_j a_{k,j}X^j \pmod{\Phi_N(X)}.$$

For $1\le k\le d$, their kth power sum is

$$\tau_k=\frac{1}{2r}\sum_j a_{k,j}c_N(j),\qquad c_N(j)=\sum_{a\in(\mathbb{Z}/N\mathbb{Z})^\times}\zeta_N^{aj}=\mu\left(\frac{N}{(N,j)}\right)\frac{\varphi(N)}{\varphi(N/(N,j))}. \tag{8.2}$$

For the first equality, each conjugate occurs $2r$ times in the full cyclotomic trace. The last is the standard Ramanujan-sum identity (group roots of unity by exact order and apply Möbius inversion). Newton's identities now give the minimal polynomial $X^d+e_1X^{d-1}+\cdots+e_d$ by

$$e_0=1,\qquad e_k=-\frac{1}{k}\sum_{j=1}^{k}e_{k-j}\tau_j\qquad(1\le k\le d). \tag{8.3}$$

We thus need only polynomial reduction and integer arithmetic.

For $q=4,8,16,32,64$, the power sums supplied by (8.2) are, respectively,

$$(-1),\quad(-3),\quad(-8,66),\quad(-22,506),\quad(-56,4164,-265256,17056004).$$

Substituting into (8.3) gives Table 1. Taking the negative roots in the quadratic cases, we obtain

$$S_4=-1,\qquad S_8=-3,\qquad S_{16}=-4-\sqrt{17},\qquad S_{32}=-11-2\sqrt{33}. \tag{8.4}$$

8.3. **Spectral scales and initial values.** Table 2 gives the two error scales and the first correction coefficient. For large q the scales nearly coincide, as predicted by (1.17) (the decimal entries are rounded evaluations of the exact formulas and are not used in the proofs).

For $r=2,3,4,5$, formula (7.10) gives, to four decimal places, the coefficient pairs (c_+,c_-)

$$(0.4268,0.0732),\quad(0.1863,0.4173),\quad(0.0724,0.7870),\quad(0.0274,1.2452).$$

TABLE 2. The first two error scales and the projective correction coefficient $c_{\mathrm{sec}} = S_q/(q+1)$. For $r = 1$, the eigenvalue -1 has coefficient zero and the only transient eigenvalue is 0.

r	q	$\lvert\lambda^{\mathrm{sec}}\rvert/2$	$\alpha_q/2$	S_q	c_{sec}
1	2	0.50000	—	0	0
2	4	0.80902	0.70711	-1	$-1/5$
3	8	0.93969	0.92388	-3	$-1/3$
4	16	0.98297	0.98079	-8.12311	-0.47783
5	32	0.99547	0.99518	-22.48913	-0.68149
6	64	0.99883	0.99880	-64.26368	-0.98867
7	128	0.99970	0.99970	-187.71509	-1.45516
8	256	0.99993	0.99992	-555.33551	-2.16084

TABLE 3. Initial values of $b(n)$. For $r \geq 4$, equation (5.10) identifies all displayed values with the algebraic-group count.

n	$r = 1$	$r = 2$	$r = 3$	$r \geq 4$
0	1	1	1	1
1	1	1	1	1
2	2	1	1	1
3	4	3	3	3
4	8	3	3	3
5	16	9	9	9
6	32	10	9	9
7	64	29	29	29
8	128	37	29	29
9	256	100	99	99
10	512	145	99	99
11	1024	363	351	351
12	2048	582	351	351
13	4096	1365	1273	1273
14	8192	2353	1274	1273

At $q = 4$, the exact pair is

$$c_+ = \frac{2+\sqrt{2}}{8}, \qquad c_- = \frac{2-\sqrt{2}}{8}.$$

The $r = 3$ column of Table 3 first differs from the algebraic-group count at $n = 14$. The new highest-weight summand $T(14)$ has the all-2 digit string and restricts to $P(\varnothing) \oplus \mathrm{St}$, giving the one additional summand.

8.4. **Odd characteristic.** For odd q, the central element $-I$ acts on the natural module by -1, giving peripheral eigenvalues 2 and -2 and a leading term that depends on parity. The nonsplit comparison also changes: its extreme noncentral value is

$$-2\cos\frac{2\pi}{q+1},$$

rather than (1.11). Donkin's tensor product theorem still gives a digit description, using tilting factors of highest weights in $\{p-1, \ldots, 2p-2\}$, but a new comparison is needed to decide whether the nonsplit value dominates the transient spectrum. We do not carry out that comparison here; the calculation above is in characteristic 2.

Appendix A. Supplementary calculations

The accompanying files implement the carry rule, construct the action matrix, evaluate the spectral coefficients, and reproduce the examples. Tensor calculations use integer arithmetic. The state rule (3.2) is implemented below, with the output mapping each resulting state to its multiplicity.

```
def add(c, j=0):
    c = list(c)
    if c[j] < 2:
        c[j] += 1
        return {tuple(c): 1}
    c[j] = 1
    out = {tuple(c): 2}
    for t, a in add(tuple(c), (j + 1) % len(c)).items():
        out[t] = out.get(t, 0) + a
    return out
```

We can calculate the projective multiplicities in (5.4) separately using the ordinary Grothendieck-ring identity $[X_2] = 2[k]+[V^{[1]}]$. The routine below computes the Steinberg composition multiplicity of a state. Choosing shifted simple terms at doubled sites reduces the total number of tensor factors, so the recursion terminates.

```
from itertools import product


def st_mult(c):
    d = [j for j, a in enumerate(c) if a == 2]
    if not d:
        return int(all(a == 1 for a in c))
    out = 0
    for bits in product((0, 1), repeat=len(d)):
        t = [int(a == 1) for a in c]
        for j, b in zip(d, bits):
            t[(j + 1) % len(c)] += b
        out += 2 ** (len(d) - sum(bits)) * st_mult(tuple(t))
    return out
```

The scripts reproduce Tables 1–3 and the matrix for $q = 8$, with direct checks of the projective multiplicities and the polynomial identities above. The high-precision numerical evaluation of the spectral expansion is separate from the integer calculations.

Author Contributions

Author order reflects relative contribution. Pranava Kumar and Taksh Kothari developed the mathematical theory, executed the proofs, and drafted the manuscript; Manolis Kellis provided computational resources and supervised the project. All authors reviewed the final manuscript.

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Competing Interests

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